# INTERMITTENCY OF PASSIVE ADVECTION†

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## 1. Introduction

An explanation of the origin of intermittency in the fully developed turbulence remains one of the main open problems of theoretical hydrodynamics. Quantitatively, intermittency is measured by deviations from Kolmogorov's scaling of the velocity correlators. Below, we shall describe a recent progress in the understanding of intermittency in a simple system describing the passive advection of a scalar quantity (temperature, density of a pollutant) by a random velocity field. The system, maintained in a stationary state by a large scale source, exhibits a down-scale energy cascade. If the molecular diffusion is small, an intermittent inertial range sets in [1]. We shall explain this phenomenon by identifying the origin of the anomalous scaling of scalar correlators in a simple model of passive advection due to Kraichnan [2].

## 2. Kraichnan model

The advection of a scalar quantity $\theta(t,\mathbf{r})$ by a velocity field $\mathbf{v}(t,\mathbf{r})$ is described by the linear differential equation

$$\partial_t\theta\,+\,\mathbf{v}\cdot\nabla\theta\,=\,f \tag{1}$$

where $f(t,\mathbf{r})$ represents the external source of the scalar. If the source vanishes then the scalar is carried along by the flow:

$$\theta(t,\mathbf{r}(t))\,=\,\theta(t_0,\mathbf{r}_0) \tag{2}$$

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where $\mathbf{r}(t)$ is the (Lagrangian) trajectory of the fluid particle located at time $t_0$ at $\mathbf{r}_0$:

$$\frac{d\mathbf{r}}{dt} \,=\, \mathbf{v}(t,\mathbf{r})\,, \qquad \mathbf{r}(t_0)=\mathbf{r}_0\,. \tag{3}$$

The non-zero source $f$ keeps creating the scalar along the Lagrangian trajectory and Eq. (2) is modified to

$$\theta(t,\mathbf{r}(t)) \,=\, \theta(t_0,\mathbf{r}_0) \,+\, \int_{t_0}^{t} ds\ f(s,\mathbf{r}(s))\,. \tag{4}$$

In the presence of diffusion of the scalar the above equations require small changes:

1. Eq. (1) picks up a term $\kappa\nabla^2\theta$ on the right hand side with $\kappa$ standing for the diffusion constant,
2. Brownian motions $\beta(t)$ should be superimposed on the Lagrangian trajectories by adding the term $\kappa\frac{d\beta}{dt}$ to the velocity in Eq. (3),
3. the right hand sides of Eqs. (2) and (4) should be averaged over $\beta$.

The Kraichnan model of the passive advection of the scalar [2] assumes velocities decorrelated at different times, and, for a fixed time, distributed as a Gaussian (i.e. non-intermittent) field with mean zero and a non-smooth typical behavior in space

$$|\mathbf{v}(t,\mathbf{r})-\mathbf{v}(t,\mathbf{r}')| \;\sim\; |\mathbf{r}-\mathbf{r}'|^{\xi/2} \tag{5}$$

where $\xi$ is a fixed parameter between 0 and 2. This may be achieved by imposing the velocity covariance

$$\Big\langle \mathbf{v}^\alpha(t,\mathbf{r})\ \mathbf{v}^\beta(t',\mathbf{r}')\Big\rangle \;=\; \delta(t-t')\ \mathcal{D}^{\alpha\beta}(\mathbf{r}-\mathbf{r}') \tag{6}$$

with

$$\mathcal{D}^{\alpha\beta}(0)-\mathcal{D}^{\alpha\beta}(\mathbf{r}) \;=\; D_0\, r^\xi \left[(d-1+\xi)\,\delta^{\alpha\beta} \;-\; \xi\,\tfrac{r^\alpha r^\beta}{r^2}\right] \;\equiv\; D^{\alpha\beta}(\mathbf{r}) \tag{7}$$

where $d$ denotes the space dimension. The typical behavior (5) follows since $D^{\alpha\beta}(\mathbf{r})\propto r^\xi$. The incompressibility $\nabla\cdot\mathbf{v}=0$ of the velocity field is assured by the relation $\partial_{r^\alpha}\mathcal{D}^{\alpha\beta}(\mathbf{r})=0$.

We shall be interested in the effect caused by a steady injection of the scalar on distances longer than some large scale $L$. One may conveniently

model such a source by a random Gaussian field $f(t,\mathbf{r})$ with mean zero and covariance

$$\langle f(t,\mathbf{r})\ f(t',\mathbf{r}')\rangle \ =\ \delta(t-t')\ \mathcal{C}(\tfrac{\mathbf{r}-\mathbf{r}'}{L}) \tag{8}$$

where $\mathcal{C}(\frac{\mathbf{r}}{L})$ is approximately constant for $r \ll L$ and decays fast for $r \gg L$. We shall assume $f$ independent of the velocities $\mathbf{v}$.

## 3. Steady state

The evolution equations of the hydrodynamical type imply identities for the correlators of the evolving quantities known under the name of Hopf equations. These equation usually couple the correlators with different number of points. The case of the passive advection is no exception since its Hopf equations couple the scalar equal-time correlators

$$F_N(t;\underline{\mathbf{r}}) \ =\ \Big\langle \prod_{n=1}^{N} \theta(t,\mathbf{r}_n)\Big\rangle \tag{9}$$

to the mixed correlators involving both the scalar and the velocity field. However, in the Kraichnan model with temporarily decorrelated velocities, the mixed correlators may be expressed by the known 2-point function of the velocities and the correlators of the scalar alone. The resulting Hopf equations may be easily seen to take the form

$$\partial_t F_N \ =\ -M_N F_N \ +\ \mathcal{C}\otimes F_{N-2} \tag{10}$$

with the shorthand notation

$$(\mathcal{C}\otimes F_{N-2})(t;\underline{\mathbf{r}}) \ =\ \sum_{n<m} \mathcal{C}(\tfrac{\mathbf{r}_n-\mathbf{r}_m}{L})\ F_{N-2}(t;\mathbf{r}_1,\underset{\hat{n}\ \hat{m}}{\ldots\ldots},\mathbf{r}_N) \tag{11}$$

and with

$$M_N \ =\ -\tfrac{1}{2}\sum_{\substack{n,m\\ \alpha\beta}} \mathcal{D}^{\alpha\beta}(\mathbf{r}_n-\mathbf{r}_m)\ \partial_{r_n^\alpha}\partial_{r_m^\beta} \tag{12}$$

being a symmetric (due to the incompressibility), positive, singular elliptic differential operator of the $2^{\rm nd}$ order. In the presence of the diffusion, $M_N$ should be modified by the subtraction of $\sum_n \kappa(\nabla_{\mathbf{r}_n})^2$. In the translationally invariant sector, i.e. for the homogeneous distributions of the scalar, the matrix $\mathcal{D}^{\alpha\beta}(\mathbf{r})$ in the definition of $M_N$ may be replaced by $-D^{\alpha\beta}(\mathbf{r})$ since the contribution of $\mathcal{D}^{\alpha\beta}(0)$ drops out. Below, we shall use this scaling form of dimension $(length)^{\xi-2}$ of the operators $M_N$ assuming also that the diffusion constant $\kappa$ has been taken to zero.

The Hopf equations (10) may be solved by induction with respect to $N$. Denoting by $P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}')$ the Green functions $\mathrm{e}^{-tM_N}(\underline{\mathbf{r}},\underline{\mathbf{r}}')$, we obtain

$$\begin{aligned} F_N(t;\underline{\mathbf{r}}) \;&=\; \int P_N(t-t_0;\underline{\mathbf{r}},\underline{\mathbf{r}}')\; F_N(t_0;\underline{\mathbf{r}}')\; d\underline{\mathbf{r}}' \\ &+\; \int\limits_{t_0}^{t} ds \int P_N(t-s;\underline{\mathbf{r}},\underline{\mathbf{r}}')\; (\mathcal{C}\otimes F_{N-2})(s,\underline{\mathbf{r}}')\; d\underline{\mathbf{r}}'\,. \end{aligned} \tag{13}$$

For a concentrated initial distribution of $\theta$ with fast decaying $N$-point functions and for $t\to\infty$, the $N$-point functions $F_N(t;\underline{\mathbf{r}})$ tend to the solution $F_N(\underline{\mathbf{r}})$ of the stationary version of the Hopf equations

$$M_N F_N \;=\; \mathcal{C}\otimes F_{N-2} \tag{14}$$

inductively determined by the relations

$$F_N(\underline{\mathbf{r}}) \;=\; \int G_N(\underline{\mathbf{r}},\underline{\mathbf{r}}')\; (\mathcal{C}\otimes F_{N-2})(\underline{\mathbf{r}}')\; d\underline{\mathbf{r}}'\,. \tag{15}$$

where $G_N(\underline{\mathbf{r}},\underline{\mathbf{r}}') = \int_0^\infty ds\; P_N(s;\underline{\mathbf{r}},\underline{\mathbf{r}}')$ is the kernel of the inverse of the operator $M_N$. In particular, the limiting stationary distribution is independent of the initial one and has vanishing odd-point functions.

## 4. Zero mode dominance

We are interested in the scaling properties of the stationary $N$-point functions $F_N(\underline{\mathbf{r}})$ for the injection scale $L$ large w.r.t. point separations, i.e. in the inertial range. Since $M_N$ have the scaling dimension of $(length)^{\xi-2}$ and $\mathcal{C}(\frac{\mathbf{r}}{L})$ is approximately constant for large $L$, we might expect that the solutions of the chain (14) of equations scale like

$$F_N(\lambda\underline{\mathbf{r}}) \;=\; \lambda^{\frac{(2-\xi)N}{2}}\; F_N(\underline{\mathbf{r}})\,. \tag{16}$$

This would be the normal Kolmogorov-Obukhov-Corrsin scaling.

In the 1994 paper [3], Kraichnan has argued in favor of the anomalous scaling of the scalar correlators. His paper steered a renewed interest in the problem which led to the discovery of a simple mechanism by which the correlators avoid the normal scaling. As was realized in [4] and [5], see also [6], for large $L$ the normally scaling contributions to $F_N$ are dominated by the ones of the *zero modes* of the operators $M_N$. The stationary Hopf equations (14) leave the freedom to add the solutions of their homogeneous version $M_N f_N = 0$. It is not obvious, however, whether the solution (15), which is defined unambiguously, contains the scaling zero modes which

become leading in the inertial range. Such zero mode contributions were, indeed, found in the 2-point function. The latter may be calculated exactly [2] with the result

$$F_2(r) \;=\; A_0\, L^{2-\xi} \;+\; A_1 r^{2-\xi} \;+\; \ldots \tag{17}$$

where the terms vanishing as inverse powers when $L \to \infty$ were omitted. The constant $\propto L^{2-\xi}$ is a zero mode of operator $M_2$ and it dominates for $r \ll L$. The constant terms, as well as the ones independent of some of $\mathbf{r}_n$'s, however, do not contribute to the structure functions

$$S_N(r) \;=\; \left\langle\, (\theta(\mathbf{r}) - \theta(\mathbf{0}))^N \right\rangle \tag{18}$$

of the scalar. As a result, the 2-point structure function scales as $r^{2-\xi}$, i.e. normally.

What about the higher-point functions? In [4] and in [5] it was shown that the 4-point structure function is dominated for $r \ll L$ by the contribution of a zero mode of the operator $M_4$ for, respectively, small velocity exponent $\xi$ and large space dimension $d$. In [7] and [8] these results have been extended to $N$-point functions with the result

$$S_N(r) \quad \propto \quad r^{\zeta_N} \qquad \text{with} \quad \zeta_N \;=\; \tfrac{(2-\xi)N}{2} \;-\; \begin{cases} \frac{N(N-2)\xi}{2(d+2)} \;+\; \mathcal{O}(\xi^2)\,, \\ \frac{N(N-2)\xi}{2d} \;+\; \mathcal{O}(d^{-2})\,. \end{cases} \tag{19}$$

The scaling dimensions of the relevant zero modes where found by restricting operators $M_N$ to scaling functions. Upon such a restriction, $M_N$'s become operators with discrete spectrum to which standard perturbative technics apply. A more difficult perturbative analysis around $\xi = 2$ [6][10][11] confirms the zero mode dominance of the inertial-range scaling also in this regime.

## 5. Slow modes

The Green functions $P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}') = \mathrm{e}^{-t\,M_N}(\underline{\mathbf{r}},\underline{\mathbf{r}}')$ have a natural interpretation in the language of Lagrangian trajectories. They are the joint probability distribution functions (PDF's) of the time $t$ positions $\mathbf{r}_n$ of the trajectories $\mathbf{r}_n(s)$ starting at time zero at points $\mathbf{r}'_n$. This follows from the $\mathcal{C} = 0$ version of Eq. (13) if we recall that, in the absence of the source, the scalar density is carried along the Lagrangian trajectories. This interpretation of $P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}')$ shows that, effectively, the (differences of the) Lagrangian trajectories undergo a simple diffusion process with the space dependent diffusion matrix $\frac{1}{2}D^{\alpha\beta}(\mathbf{r}_n - \mathbf{r}_m) \;\propto\; |\mathbf{r}_n - \mathbf{r}_m|^\xi$ so that they diffuse

very slowly when they are close but faster and faster when, eventually, they separate.

Let us look closer at the effective diffusion of the Lagrangian trajectories. The scaling property of operators $M_N$ implies that

$$P_N(\lambda^{2-\xi}t;\lambda\underline{\mathbf{r}},\lambda\underline{\mathbf{r}}_0)\ =\ \lambda^{-Nd}\,P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}_0)\,. \tag{20}$$

Thus the *distances* scale as $(time)^{\frac{1}{2-\xi}}$ as compared to $(time)^{\frac{1}{2}}$ in the standard diffusion. We may probe the effective spread of Lagrangian trajectories with scaling functions $f_N$ such that $f_N(\lambda\underline{\mathbf{r}})=\lambda^\sigma f_N(\underline{\mathbf{r}})$, e.g. with the sum of squares of distances between $\mathbf{r}_n$'s. The average of $f_N$ over the time zero positions of $N$ Lagrangian trajectories which at time $t$ pass by points $\underline{\mathbf{r}}$ is[1]

$$\begin{aligned}\langle\, f_N\rangle_{t,\underline{\mathbf{r}}}\ &=\int P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}')\ f_N(\underline{\mathbf{r}}')\ d\underline{\mathbf{r}}'\ =\ \lambda^{-\sigma}\!\int\! P_N(\lambda^{2-\xi}t;\lambda\underline{\mathbf{r}},\underline{\mathbf{r}}')\ f_N(\underline{\mathbf{r}}')\ d\underline{\mathbf{r}}'\\ &=\ (\tfrac{t}{\tau})^{\frac{\sigma}{2-\xi}}\int P_N(\tau;(\tfrac{\tau}{t})^{\frac{1}{2-\xi}}\underline{\mathbf{r}},\ \underline{\mathbf{r}}')\ f_N(\underline{\mathbf{r}}')\ d\underline{\mathbf{r}}' \end{aligned}\tag{21}$$

where we have used the scaling properties of $f_N$ and $P_N$ and have set $\tau=\lambda^{2-\xi}t$. If we let $t$ grow and keep $\tau$ constant, the last integral may be expected to tend to the limit $\int P_N(\tau;0,\underline{\mathbf{r}}')\,f_N(\underline{\mathbf{r}}')\,d\underline{\mathbf{r}}'$ so that, if the latter integral does not vanish,

$$\langle\, f_N\rangle_{t,\underline{\mathbf{r}}}\quad\propto\quad(\tfrac{t}{\tau})^{\frac{\sigma}{2-\xi}} \tag{22}$$

This is the super-diffusive behavior which sets in for typical scaling probes $f_N$. Suppose, however, that $f_N$ is a scaling zero mode of $M_N$. Since

$$\frac{d}{dt}\,\langle\, f_N\rangle_{t,\underline{\mathbf{r}}}\ =\ -\int P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}')\ (M_Nf_N)(\underline{\mathbf{r}}')\ d\underline{\mathbf{r}}'\ =\ 0\,, \tag{23}$$

the average $\langle\, f_N\rangle_{t,\underline{\mathbf{r}}}$ is constant in time instead of growing like $(\frac{t}{\tau})^{\frac{\sigma}{2-\xi}}$: the zero modes describe structures preserved in mean by the Lagrangian flow.

A constant or linear functions are the obvious examples of scaling zero modes but a closer analysis [12] of operators $M_N$ shows that there is a whole discrete series $f_{N,i}$ of them with scaling dimensions $\sigma_i\geq 0$. Besides, each such scaling zero mode gives rise to a tower of *slow modes* $f_{N,ik}$, $k=0,1,\dots$, starting at $f_{N,i0}=f_{N,i}$, with scaling dimensions $\sigma_i+(2-\xi)k$. The averages $\langle\, f_{N,ik}\rangle_{t,\underline{\mathbf{r}}}$ are polynomials of order $k$ in $t$ so that they grow

[1] for the later convenience, we study the backward evolution in time

slower than the typical behavior (22). All these scaling modes appear in the small $\lambda$ asymptotic expansion [12]

$$P_N(\tau;\lambda\underline{\mathbf{r}},\underline{\mathbf{r}}') \;=\; \sum_{i,k}\lambda^{\sigma_i+(2-\xi)k}\; f_{N,ik}(\underline{\mathbf{r}})\; g_{N,ik}(\tau;\underline{\mathbf{r}}') \tag{24}$$

which, when inserted on the right hand side of Eq. (21), gives

$$\langle\, f_N\rangle_{t,\underline{\mathbf{r}}} \;=\; \sum_{i,k}\left(\tfrac{t}{\tau}\right)^{\frac{\sigma-\sigma_i}{2-\xi}-k}\; f_{N,ik}(\underline{\mathbf{r}})\;\int g_{N,ik}(\tau;\underline{\mathbf{r}}')\; f_N(\underline{\mathbf{r}}')\; d\underline{\mathbf{r}}'\,. \tag{25}$$

The functions $g_{N,ik}(\tau;\underline{\mathbf{r}})$ are finite and decay fast for large $\underline{\mathbf{r}}$. For generic $f_N$, the constant zero mode $f_{N,00}$ corresponding to $g_{N,00}(\tau;\underline{\mathbf{r}}')=P_N(\tau;\mathbf{0},\underline{\mathbf{r}}')$ gives the dominant contribution leading to the behavior (22). This term (and many others) vanishes for $f_N=f_{N,jl}$ resulting in the slower growth.

Upon integration over $t$, the asymptotic expansion (24) induces the one for the kernels of the inverses of $M_N$'s:

$$G_N(\lambda\underline{\mathbf{r}},\underline{\mathbf{r}}') \;=\; \sum_{i}\lambda^{\sigma_i}\; f_{N,i}(\underline{\mathbf{r}})\; h_{N,i}(\underline{\mathbf{r}}') \tag{26}$$

(the contributions of the slow modes disappear under the time integration). This expansion is behind the zero-mode dominance of the short-distance behavior of the iterative solutions (15) for the stationary $N$-point functions of the scalar. The subtractions required in the passage from the correlation functions to the structure functions leave only the contributions of the zero modes $f_{N,i}$ fully dependent on all $\mathbf{r}_n$'s.

## 6. Fuzzy trajectories

The asymptotic expansion (24) describes what happens when the final points of the Lagrangian trajectories are taken together. It was established with mathematical rigor for the case of two Lagrangian trajectories and was checked in perturbative approaches for $N>2$. At the first glance, it may look bizarre. Indeed, we may naively expect that if the final points $\mathbf{r}_n$ of the trajectories converge then their initial points should also do with the joint PDF $P_N(t;\lambda\underline{\mathbf{r}},\underline{\mathbf{r}}')$ becoming proportional to $\prod\delta(\mathbf{r}'_n-\mathbf{r}'_{n+1})$ when $\lambda\to 0$. Instead, $P_N(t;\lambda\underline{\mathbf{r}},\underline{\mathbf{r}}')$ tends to a regular limit $g_{N,00}(t;\underline{\mathbf{r}}')$. The mechanism by which $P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}')$ avoids the singularity at $\underline{\mathbf{r}}=0$ is somewhat subtle [12]. Recall that the Lagrangian trajectories satisfy the differential equation (3). The uniqueness of solutions of such an equation requires the Lipschitz condition $|\mathbf{v}(t,\mathbf{r})-\mathbf{v}(t,\mathbf{r}')|\sim|\mathbf{r}-\mathbf{r}'|$ for $\mathbf{r}'$ tending to $\mathbf{r}$. But our velocities are only Hölder continuous in $\mathbf{r}$ with exponent $\frac{\xi}{2}<1$, see Eq. (5). The resulting non-uniqueness of the Lagrangian trajectories passing through a fixed

point violates the Newton-Leibniz paradigm and allows for a continuum of trajectories with coincident final points. Although the trajectories keep collapsing continuously, the probabilistic quantities such as the joint PDF's $P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}')$ still make perfect sense. They reflect, however, the fuzzyness of the trajectories in their asymptotics (24).

A less rigorous-minded reader might object that the non-uniqueness of the Lagrangian trajectories is a mathematical nuisance since more realistic turbulent velocities, even when showing the behavior (5) with $\xi$ close to $\frac{2}{3}$ in the inertial range, are smoothed on short distances by the viscous effects so that their Lagrangian trajectories are unique. A closer examination, shows, however, that such Lagrangian trajectories, although uniquely determined by sharp values at one time, exhibit a *sensitive dependence* on these values signaled by non-zero *Lyapunov exponents*. As a result, if the fixed-time positions of the trajectories have an $\epsilon$-spread, the trajectories at different times are spread in a large region and will stay such if the viscosity is taken to zero first and the spread $\epsilon$ only next. The non-uniqueness of the trajectories caused by their continuous collapse is simply a useful mathematical abstraction describing a real physical phenomenon: a fast spread of the trajectories in each realization of the high Reynolds number velocity field. Such a spread makes the identification of the trajectories practically impossible. It should be remarked that in the incompressible case, the collapse of the trajectories must be accompanied symmetrically by their branching since $P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}') = P_N(t;\underline{\mathbf{r}}',\underline{\mathbf{r}})$. In compressible flows, this symmetry is broken. In particular, in the utmost compressible Burgers flows, the trajectories only collapse together (in a discrete process) sticking to the shocks.

## 7. Numerical results

The region of $\xi$ neither close to 0 nor to 2 has up to now defied an exact analysis. The first numerical simulations of the system [13][14] were based on the direct solution of the scalar equation (1). They did not cover the region of small $\xi$ and did not permit to discriminate between the small ($\propto \xi$) anomalous structure-function exponents implied by the perturbative zero-mode analysis and the different ($\propto 1$) predictions of a closure of the structure-function equations proposed in [3], see also [15].

Recently, a new numerical analysis [16] of the 4-point function of the scalar in two and three space dimensions has been performed for different values of $\xi$. It was based on a direct generation of samples of Lagrangian trajectories passing at time $t$ by $N$ fixed points. Such samples allow a direct simulation of the PDF's $P_N(t;\underline{\mathbf{r}},\underline{\mathbf{r}}')$ as well as direct simulations of the stationary correlation functions $F_N$. Let us explain how the latter were

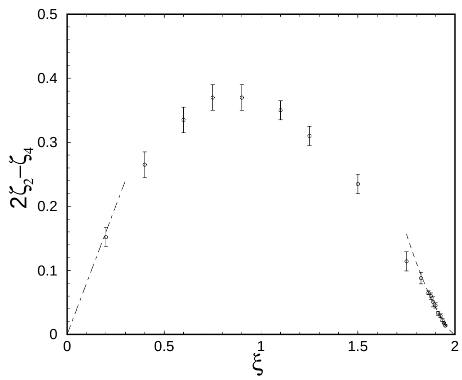


*Figure 1.* Anomalous exponent of the 4-point structure function of the scalar as a function of $\xi$ (U. Frisch, A. Mazzino and M. Vergassola, cond-mat/9802192)

done in [16]. Upon setting $t_0 = -\infty$ and $\theta(t_0) = 0$ in Eq. (4), one obtains for even $N$ the relations

$$F_N(\underline{\mathbf{r}}) \;=\; \Big\langle \prod_{n=1}^{N} \int_{-\infty}^{t} ds\; f(s, \mathbf{r}_n(s)) \Big\rangle \;=\; \sum_{\Pi} \Big\langle \prod_{\{n,m\}\in\Pi} \mathcal{T}(\mathbf{r}_n, \mathbf{r}_m) \Big\rangle \tag{27}$$

where the sum, resulting from the average over the Gaussian source, is over the pairings $\Pi$ of the indices $1, \ldots, N$ and where

$$\mathcal{T}(\mathbf{r}_n, \mathbf{r}_m) \;=\; \int_{-\infty}^{t} ds\; \mathcal{C}\big(\tfrac{\mathbf{r}_n(s)-\mathbf{r}_m(s)}{L}\big) \tag{28}$$

is approximately equal to the time spent within distance $L$ by the two trajectories with the end-points $\mathbf{r}_n$ and $\mathbf{r}_m$. The average of the product of such times may be simulated using the ensemble of trajectories passing through points $\mathbf{r}_n$. It is finite since the trajectories effectively separate as $(time)^{\frac{1}{2-\xi}}$. Due to the limited dependence of trajectories on the initial conditions discussed above, this remains true even if $\mathbf{r}_n$ and $\mathbf{r}_m$ tend to each other. The comparison of $F_N(\underline{\mathbf{r}})$ calculated this way for different configurations of points $\mathbf{r}_n$ showed the dominance of the structure functions $S_N(r)$ by subleading terms of $F_N(\underline{\mathbf{r}})$, as in the discussion at the end of Section 5. The resulting values of the anomalous 4-point-structure-function exponent $2\zeta_2 - \zeta_4$, presented in Fig. 1 borrowed from [16], confirm the predictions of the perturbative zero-mode analysis around $\xi = 0$ indicated in Fig. 1 by the broken-dotted line.

## 8. Conclusions

We have exhibited a mechanism behind intermittency of the scalar structure functions in the Kraichnan model of passive advection. The important point is that the anomalous scaling originated from a discrete series of solutions of the homogeneous Hopf equations without the forcing and diffusion terms. This, and the relation of such solutions to the effective fuzzyness of Lagrangian trajectories at high Reynolds numbers promise to be the stories that persist in other intermittent hydrodynamical systems. In particular, it has been indicated recently [17] how discrete solutions of the unforced Hopf equations may naturally give rise to a multiscaling picture of anomalous exponents in the Navier-Stokes case.

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